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A LICHNEROWICZ ESTIMATE FOR THE FIRST EIGENVALUE OF CONVEX DOMAINS IN KÄHLER MANIFOLDS

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ABSTRACT. In this article, we prove a Lichnerowicz estimate for a compact *convex* domain of a Kähler manifold whose Ricci curvature satisfies $\mathrm{Ric} \geq k$ for some constant $k > 0$. When equality is achieved, the boundary of the domain is totally geodesic and there exists a nontrivial holomorphic vector field.

We show that a ball of sufficiently large radius in complex projective space provides an example of a *strongly pseudoconvex* domain which is *not convex*, and for which the *Lichnerowicz estimate* fails.

1. INTRODUCTION

Let (M^n, g) be a compact n-dimensional Riemannian manifold. Assume first that M has no boundary. A theorem of Lichnerowicz [Lic58] asserts that if the Ricci curvature Ric of M satisfies $\mathrm{Ric} \geq k$ for some constant $k > 0$, then the first nonzero eigenvalue λ of the Laplace operator satisfies

$$\lambda \geq \frac{n}{n-1} k. \tag{1.1}$$

Here, $nk/(n-1)$ should be viewed as the first nonzero eigenvalue of the round n-dimensional sphere $S^n(k/(n-1))$ of constant curvature $k/(n-1)$. Moreover, by a result of Obata [Oba62], the equality case in (1.1) is obtained if and only if M is isometric to this sphere. Reilly considered a similar problem, but for compact manifolds with boundary [Rei77]. Namely, he proved that if M is as in Lichnerowicz theorem, except that it has now a boundary such that its mean curvature with respect to the outward normal vector field is nonnegative, then the first eigenvalue λ of the Laplace operator with the Dirichlet boundary condition still satisfies (1.1). He also proved that the equality case characterizes a hemisphere in $S^n(k/(n-1))$.

In another direction, Lichnerowicz showed that for Kähler manifolds, his estimate (1.1) can be improved. To explain this, we modify slightly our normalization conventions: we consider a closed Kähler manifold M of *real* dimension $n = 2m$, whose Ricci curvature satisfies $\mathrm{Ric} \geq k > 0$. On a Kähler manifold, there are *a priori* three Laplace operators, namely the usual Laplace operator associated to exterior differentiation, and two Laplace operators associated to ∂ and $\bar{\partial}$ respectively. Moreover, the latter two are actually equal and coincide with half the usual Laplacian. In the sequel, we

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will consider the Laplace operator associated with $\bar\partial$, and we will denote it by Δ, so that

$$\Delta = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*.$$

The first nonzero eigenvalue of Δ acting on functions will be denoted by λ. In view of these conventions, the condition $\mathrm{Ric} \geq k$ and the Lichnerowicz estimate (1.1) give

$$\lambda \geq \frac{mk}{(2m-1)}.$$

Lichnerowicz improved this bound by showing that

$$\lambda \geq k$$

Moreover, if equality is achieved, then there is a non trivial holomorphic vector field on M.

The purpose of this note is to consider the case of compact Kähler manifolds with boundary. As in Reilly's result, we will have to impose some convexity property on the boundary:

Theorem 1.1. *Let M be a compact convex domain in a Kähler manifold. Assume that the Ricci curvature satisfies* $\mathrm{Ric} \geq k$ *for some constant* $k > 0$*. Then the first nonzero eigenvalue λ of the Laplacian with Dirichlet boundary condition satisfies*

$$\lambda \geq k.$$

Moreover, if equality is achieved, then the boundary ∂M is totally geodesic and there is a nontrivial holomorphic vector field on M.

Remark 1.2. As we will see in the proof, the convexity hypothesis may be relaxed into another condition of mean curvature type (see condition (4.1) below). However, this condition has no clear geometrical meaning, so that we have stated our theorem with the convexity hypothesis instead.

Remark 1.3. It is natural to ask whether our result remains true if one assumes the pseudoconvexity of the boundary instead of its convexity. It turns out that a ball of sufficiently large radius in complex projective space provides an example of a strongly pseudoconvex domain which is not convex, and for which the Lichnerowicz estimate fails (see Proposition 4.1 for more details on this).

Remark 1.4. In the real setting, one can consider the Laplacian with the Neumann boundary condition, and again with the convexity condition, one can show that the Lichnerowicz estimate (1.1) still holds for the first nonzero eigenvalue [PMYC86]. In the Kähler setting, by using the method of proof of Theorem 1.1, it should also be possible to prove that the conclusion of this theorem is true for the first nonzero eigenvalue of the $\bar\partial$-Laplacian with the absolute $\bar\partial$-condition on the boundary.

An immediate consequence of our theorem is

Corollary 1.5. *Assume that M is a strongly convex domain in a complex manifold which can be endowed with a Kähler metric whose Ricci curvature satisfies* $\mathrm{Ric} \geq k$ *for some constant* $k > 0$*. Then the first nonzero eigenvalue λ of the Laplacian with Dirichlet boundary conditions satisfies*

$$\lambda > k.$$

Our proof follows the same strategy as in the original proofs of Lichnerowicz and Reilly. We use an appropriate Bochner formula for the Laplacian acting on $(0,1)$-forms and apply it to $\overline{\partial} f$, where the function f is an eigenfunction of Δ for the first eigenvalue. After integrating the result on M and integrating by parts, the desired eigenvalue estimate follows if we can prove that some boundary term is nonpositive, which is the case under the convexity hypothesis.

2. Background material

In this section, we recall some well-known facts that will be used in the proof of our main result.

2.1. **Decomposition of the Hessian.** Let f be a real valued smooth function on a Kähler manifold (M, J, g). Its Riemannian Hessian $\operatorname{Hess} f$ is as usual defined by

$$\operatorname{Hess} f = \nabla df$$

This Hessian may be decomposed as the sum of a J-symmetric bilinear form and a J-skew-symmetric bilinear form. More specifically, we have

$$\operatorname{Hess} f = H^1 f + H^2 f$$

where for tangent vectors A and B,

$$H^1 f(A,B) = \frac{1}{2}\{\operatorname{Hess} f(A,B) + \operatorname{Hess} f(JA,JB)\}$$

and

$$H^2 f(A,B) = \frac{1}{2}\{\operatorname{Hess} f(A,B) - \operatorname{Hess} f(JA,JB)\}$$

The two following facts may easily be checked:

(1) The $(1,1)$-form associated to $H^1 f$ by the complex structure J is $i\partial\bar{\partial} f$:
$$H^1 f(JA,B) = i\partial\bar{\partial} f(A,B).$$
(2) In local coordinates, $H^2 f$ has the following components
$$(H^2 f)_{pq} = \overline{(H^2 f)_{\bar{p}\bar{q}}} = \frac{\partial^2 f}{\partial z_p \partial z_q} - \Gamma^r_{pq}\frac{\partial f}{\partial z_r},$$
and the other components vanish. $H^2 f$ is called the *complex Hessian.*

2.2. **Bochner formula for the Laplacian.** Let (M, g) be a Kähler manifold, and denote by D its Levi-Civita connection. If α is a $(0,1)$-form, we denote by $D''\alpha$ the $(0,2)$-part of $D\alpha$. More precisely, $D\alpha$ is a section of the bundle $T^*M \otimes (T^*)^{0,1}M$; this bundle decomposes as a direct sum

$$((T^*)^{1,0}M \otimes (T^*)^{0,1}M) \oplus ((T^*)^{0,1}M \otimes (T^*)^{0,1}M),$$

and $D''\alpha$ is the projection of $D\alpha$ on the second factor of this decomposition. In local complex coordinates, we have

$$(D''\alpha)_{\bar{p}\bar{q}} = \frac{\partial \alpha_{\bar{q}}}{\partial \bar{z}_p} - \Gamma^{\bar{r}}_{\bar{p}\bar{q}}\alpha_{\bar{r}}.$$

Let now $(D'')^*$ be the formal adjoint of D''. For a section β of $(T^*)^{0,1}M \otimes (T^*)^{0,1}M$ one can see that locally

$$((D'')^*\beta)_{\bar{p}} = -g^{q\bar{r}} \frac{\partial \beta_{\bar{r}\bar{p}}}{\partial z_q}.$$

Then we have the following Bochner formula for the $\bar{\partial}$-Laplacian Δ acting on $(0,1)$-forms:

$$\Delta = (D'')^* D'' + \mathrm{Ric}\,. \tag{2.1}$$

For future reference, we also give the integration by parts formula for D'' in the presence of a boundary (see e.g. [Tay96, Proposition 9.1]). Here, we assume that M is compact, and we let n denote the outward unit normal vector field on ∂M. The $(0,1)$ part of the dual 1-form ν corresponding to n by the metric will be denoted by $\nu^{0,1}$. Finally, we let σ denote the measure induced on the boundary by the metric. For smooth α and β, we then have

$$\left\langle D''\alpha, \beta \right\rangle_{L^2(M)} = \left\langle \alpha, (D'')^*\beta \right\rangle_{L^2(M)} + \int_{\partial M} \left\langle \nu^{0,1} \otimes \alpha, \beta \right\rangle \sigma. \tag{2.2}$$

3. Estimate of the first eigenvalue

In this section, M is a compact smooth domain in a Kähler manifold of complex dimension m, with metric g and Ricci curvature bounded from below by some positive constant k. The outward unit normal vector field on the boundary ∂M is denoted by n and its covariant associated 1-form by ν.

3.1. **Bochner formula and the first eigenvalue.** Let f be a real valued eigenfunction of Δ corresponding to the first nonzero eigenvalue λ of Δ, so that $f : \overline{M} \to \mathbb{R}$ is smooth, vanishes on the boundary ∂M, and satisfies $\Delta f = \lambda f$. (Note that it is possible to choose f to be real valued, because Δ is equal to half the usual Laplacian.) We write the Bochner formula (2.1) for the $(0,1)$-form $\bar{\partial} f$ and take the L^2-inner product of the resulting equality with $\bar{\partial} f$ itself:

$$\left\langle \Delta\bar{\partial} f, \bar{\partial} f \right\rangle_{L^2(M)} = \left\langle (D'')^* D'' \bar{\partial} f, \bar{\partial} f \right\rangle_{L^2(M)} + \int_M \mathrm{Ric}\,(\bar{\partial} f, \bar{\partial} f). \tag{3.1}$$

Using the fact that $\Delta\bar{\partial} = \bar{\partial}\Delta$ and that $f|_{\partial M} = 0$, we can integrate by parts the left hand side of (3.1) to get

$$\begin{aligned}
\left\langle \Delta\bar{\partial} f, \bar{\partial} f \right\rangle_{L^2(M)} &= \left\langle \bar{\partial}\Delta f, \bar{\partial} f \right\rangle_{L^2(M)} \\
&= \left\langle \bar{\partial}(\lambda f), \bar{\partial} f \right\rangle_{L^2(M)} \\
&= \lambda \left\langle \Delta f, f \right\rangle_{L^2(M)} \\
&= \lambda^2 \left\| f \right\|^2_{L^2(M)}.
\end{aligned}$$

We can deal with the Ricci term in the right hand side of (3.1) in a similar way

$$\begin{aligned}
\int_M \mathrm{Ric}\,(\bar{\partial} f, \bar{\partial} f) &\geq k \left\langle \bar{\partial} f, \bar{\partial} f \right\rangle_{L^2(M)} \\
&= k \left\langle \Delta f, f \right\rangle_{L^2(M)} \\
&= k\lambda \left\| f \right\|^2_{L^2(M)}.
\end{aligned}$$

Finally, we can integrate by parts the first term in the right hand side of (3.1) (see formula (2.2)):

$$\left\langle (D'')^* D''\bar\partial f, \bar\partial f\right\rangle_{L^2(M)} = \left\| D''\bar\partial f\right\|^2_{L^2(M)} - \int_{\partial M} \left\langle D''\bar\partial f, \nu^{0,1}\otimes\bar\partial f\right\rangle \sigma, \tag{3.2}$$

and combining this with our previous estimates, we obtain

$$\lambda(\lambda-k)\left\|f\right\|^2_{L^2(M)} \geq \left\| D''\bar\partial f\right\|^2_{L^2(M)} - \int_{\partial M} \left\langle D''\bar\partial f, \nu^{0,1}\otimes\bar\partial f\right\rangle \sigma. \tag{3.3}$$

As a consequence, if we set

$$I = -\int_{\partial M} \langle D''\bar\partial f, \nu^{0,1}\otimes\bar\partial f\rangle\, \sigma,$$

we will get $\lambda \geq k$ provided we can prove that $I \geq 0$. In the next subsection, we will see that this is indeed the case under suitable assumptions on the boundary.

3.2. **Boundary term.** To estimate the boundary term I, we first notice that as f is real valued, we have

$$(D''\bar\partial f)_{\bar p\bar q} = (H^2 f)_{\bar p\bar q}$$

so that

$$I = -\int_{\partial M} \left\langle H^2 f, \nu^{0,1}\otimes\bar\partial f\right\rangle \sigma = -\int_{\partial M} H^2 f(n^{0,1}, (\partial f)^\sharp)\,\sigma.$$

We then choose a boundary defining function ρ for ∂M. This means that ρ is a smooth real valued function such that $M = \{\rho \leq 0\}$, $\partial M = \{\rho = 0\}$ and $d\rho$ does not vanish on ∂M. By multiplying ρ by a suitable smooth positive function if necessary, we may assume that

$$n = \operatorname{grad}\rho.$$

Moreover, near a fixed (but arbitrary) point of the boundary ∂M, we fix a local orthonormal frame adapted to the complex structure J which has the form

$$v_1, Jv_1, \dots, v_m, Jv_m = n = \operatorname{grad}\rho.$$

We also set

$$e_p = \frac{1}{\sqrt 2}(v_p - iJv_p), \quad p = 1,\dots,m.$$

Note that as f vanishes on ∂M, its derivatives along tangent vectors to ∂M also vanish and consequently

$$(\partial f)^\sharp = \frac{-i}{\sqrt 2}(n\cdot f)\bar e_m, \qquad n^{0,1} = \frac{-i}{\sqrt 2}\bar e_m,$$

where $n\cdot f$ means $df(n)$. Therefore,

$$I = \frac12\int_{\partial M} (n\cdot f)\operatorname{Hess} f(\bar e_m, \bar e_m)\,\sigma$$

which can be decomposed as $I = I_1 + iI_2$ with

$$I_1 = \frac14 \int_{\partial M} (n\cdot f)\left[\operatorname{Hess} f(Jn, Jn) - \operatorname{Hess} f(n,n)\right]\sigma$$

and

$$I_2 = \frac{-1}{2}\int_{\partial M}(n\cdot f)\,\mathrm{Hess}\, f(Jn,n)\,\sigma.$$

Actually I_2 vanishes because I is a real number. (This follows from the fact that in equation (3.1), the left hand side and the Ricci term are real numbers, so that the term involving D'' is also a real number. This implies, by equation (3.2), that the boundary term I is a real number as well. There is also a more conceptual reason for the vanishing of I_2, see section 3.3). We now turn our attention to I_1. As $\triangle f = \lambda f = 0$ on ∂M, the trace of $\mathrm{Hess}\, f$ is also zero on ∂M:

$$\mathrm{Hess}\, f(Jn,Jn) - \mathrm{Hess}\, f(n,n) = \sum_{k=1}^{m-1}\left[\mathrm{Hess}\, f(v_k,v_k) + \mathrm{Hess}\, f(Jv_k,Jv_k)\right] + 2\,\mathrm{Hess}\, f(Jn,Jn).$$

We notice that all vectors appearing in the right hand side are tangent to the boundary. For such a vector u, we have on ∂M

$$\begin{aligned}\mathrm{Hess}\, f(u,u) &= -\langle \nabla_u u, n\rangle\,(n\cdot f)\\ &= \langle \nabla_u n, u\rangle\,(n\cdot f)\\ &= (n\cdot f)\,\mathrm{Hess}\,\rho(u,u).\end{aligned}$$

This implies

$$I_1 = \frac{1}{4}\int_{\partial M}(n\cdot f)^2\Big\{\sum_{k=1}^{m-1}\left[\mathrm{Hess}\,\rho(v_k,v_k) + \mathrm{Hess}\,\rho(Jv_k,Jv_k)\right] + 2\,\mathrm{Hess}\,\rho(Jn,Jn)\Big\}\,\sigma. \tag{3.4}$$

If we assume that ∂M is convex, then all terms in the integrand of the right hand side are nonnegative, so that $I = I_1 \geq 0$ as desired. This proves that $\lambda \geq k$ in the convex case.

It remains to deal with the equality case. If we assume that $\lambda = k$, then by equation (3.3), we must have $D''\bar\partial f = 0$ and $I = 0$. On the one hand, $D''\bar\partial f = 0$ means that the $(1,0)$-vector field associated to $\bar\partial f$ by the metric is a (nonzero) holomorphic vector field. On the other hand, from $I = 0$, we infer that the integrand in equation (3.4) has to vanish identically on the boundary:

$$(n\cdot f)^2\Big\{\sum_{k=1}^{m-1}\{\mathrm{Hess}\,\rho(v_k,v_k) + \mathrm{Hess}\,\rho(Jv_k,Jv_k)\} + 2\,\mathrm{Hess}\,\rho(Jn,Jn)\Big\} = 0.$$

Assume by contradiction that ∂M is not totally geodesic (but still convex of course). Then the term between the brackets is positive at some point and we will get the vanishing of $n.f$ on an open subset of ∂M. But f is in the kernel of the elliptic operator $\Delta - \lambda$ and vanishes on ∂M. By the unique continuation principle for elliptic operators (see e.g. [BBW93]), f has to vanish on M as well, which is absurd. Therefore, ∂M is totally geodesic. This completes the proof of Theorem 1.1.

3.3. **A direct proof that the boundary term is real.** The fact that

$$I_2 = \frac{-1}{2}\int_{\partial M} (n\cdot f)\,\mathrm{Hess}\, f(Jn, n)\,\sigma$$

vanishes is also a consequence of the fact that the expression

$$(n\cdot f)\,\mathrm{Hess}\, f(Jn, n)\,\sigma = (n\cdot f)(Jn\cdot n\cdot f)\,\sigma$$

is an exact differential form on the closed manifold ∂M. Indeed, the vector field $Jn = J\,\mathrm{grad}\,\rho$ is the Hamiltonian vector field associated to ρ. This means that if ω is the Kähler form, then

$$i_{Jn}\,\omega = -d\rho.$$

Hence

$$d\, i_{Jn}\, i_n\, \omega^m = -md(n\cdot\rho)\wedge\omega^{m-1} - m(m-1)d\rho\wedge d\, i_n\,\omega\wedge\omega^{m-2}$$

Let $j : \partial M \to M$ be the inclusion map. Since the functions $n\cdot\rho$ and ρ are constant on ∂M, we have

$$j^*\left(d\, i_{Jn}\, i_n\, \omega^m\right) = 0.$$

Now, Jn is a vector field defined on a neighborhood of ∂M whose restriction to ∂M is tangent to ∂M, so that

$$j^*(i_{Jn}\,\beta) = i_{Jn} j^*(\beta)$$

for any differential form β. As a consequence, we get

$$d\, i_{Jn} j^*\left(i_n\,\omega^m\right) = 0.$$

Finally, we have

$$j^*\left(i_n\,\omega^m\right) = \sigma,$$

and

$$d\, i_{Jn}\sigma = 0.$$

Defining a vector field X by

$$X = \frac{1}{2}(n\cdot f)^2 Jn,$$

it follows that on ∂M, we have

$$d\, i_X\,\sigma = (n\cdot f)(Jn\cdot n\cdot f)\sigma.$$

4. Counter-example in the pseudoconvex case

We use the notation introduced in the previous section. It is clear from the proof of Theorem 1.1 that in order to get the estimate $\lambda \geq k$, it is enough to assume that on the boundary we have

$$\sum_{k=1}^{m-1}\{\mathrm{Hess}\,\rho(v_k, v_k) + \mathrm{Hess}\,\rho(Jv_k, Jv_k)\} + 2\,\mathrm{Hess}\,\rho(Jn, Jn) \geq 0, \tag{4.1}$$

and not necessarily the convexity of ∂M. We may rewrite this condition as

$$\sum_{k=1}^{m-1} H^1\rho(v_k, v_k) + \mathrm{Hess}\,\rho(Jn, Jn) \geq 0.$$

Here, $\sum_{k=1}^{m-1} H^1\rho(v_k, v_k)$ is the trace of the Levi form of the boundary, which would be nonnegative if ∂M were assumed to be only pseudoconvex. The extra term $\operatorname{Hess}\rho(Jn, Jn)$, however, can usually not be controlled in the pseudoconvex case. This suggests that the conclusion of Theorem 1.1 does not generally hold in this case, as we now explain.

We consider here the complex m-dimensional projective space $\mathbb{P}^m(\mathbb{C})$ equipped with the Fubini-Study metric normalized so that the holomorphic sectional curvature is 4 (the Einstein constant is thus $2(m+1)$ and the diameter is $\pi/2$).

Proposition 4.1. *Fix some point $x \in \mathbb{P}^m(\mathbb{C})$, some $r_0 \in]0, \pi/2[$, and let M be the geodesic ball centered at x, of radius r_0.*

(i) *If $r_0 \in]\pi/4, \pi/2[$, then M is strongly pseudoconvex, not convex.*
(ii) *The first nonzero eigenvalue of M with Dirichlet boundary conditions goes to 0 as r_0 approaches $\pi/2$*

Proof. The first point is a well-known result. For completeness, we outline the proof here. Denote by r the distance function from x, and set $\rho = r^2 - r_0^2$, so that ρ is a smooth defining function for M. We want to compute the eigenvalues of the Hessian of ρ. As

$$\operatorname{Hess}\rho = 2r \operatorname{Hess} r + 2dr \otimes dr,$$

we will only have to compute the eigenvalues of $\operatorname{Hess} r$. To do this, we proceed as in the proof of [GW79, Theorem A, p.19]. Recall that for a tangent vector u, the curvature $R(u, .)u$ of $\mathbb{P}^m(\mathbb{C})$ is given by ([BGM71, Proposition F.34])

$$R(u, .)u = \begin{cases} 0, & \text{on } \mathbb{R}u; \\ 4\mathrm{Id}, & \text{on } \mathbb{R}Ju; \\ \mathrm{Id}, & \text{on the orthogonal complement of } (u, Ju). \end{cases}$$

Let γ be a normal geodesic starting from x. We can choose a parallel frame along γ which has the form $v_1, Jv_1, \dots, v_m, Jv_m = \operatorname{grad} r$. Using the explicit expression of R, it is then easy to check that the space of Jacobi fields V along γ satisfying $V(0) = 0$ and $V \perp \dot\gamma$ has as basis $V_i = \sin(r)v_i$, JV_i, $i = 1, \dots, m-1$ and $V_m = \sin(2r)v_m$. Using the second variation formula, we see that $\operatorname{Hess} r$ is diagonalized in the basis $v_1, Jv_1, \dots, v_m, Jv_m$ with eigenvalues $\cot(r)$ (of order $2m-2$), $2\cot(2r)$ and 0. If $r = r_0 \in]\pi/4, \pi/2[$, we infer that the Levi form of ρ is positive definite, being equal to $2r_0 \cot(r_0)\mathrm{Id}$ on the Levi distribution. In other words, M is strongly pseudoconvex. However, M is not convex because the principal curvature $2\cot(2r_0)$ is negative.

As for the second point of our proposition, it is for example a consequence of a more general result due to Chavel and Feldman [CF78, Theorem 1] which states the following: let X be a compact Riemannian manifold and let $X' \subset X$ be a submanifold. For small $\varepsilon > 0$, let X'_ε be the ε-neighborhood of X' in X and denote by Ω_ε the set $X \setminus X'_\varepsilon$. Let (λ_j) be the spectrum of X and let $(\lambda_j(\varepsilon))$ be the spectrum of Ω_ε with Dirichlet boundary conditions. If the codimension of X' in X is at least 2, then for all j, $\lambda_j(\varepsilon) \to \lambda_{j-1}$ as $\varepsilon \to 0$. In our case, we can take $X = \mathbb{P}^m(\mathbb{C})$ and $X' = \mathbb{P}^{m-1}(\mathbb{C})$ which we view as the cut locus of our fixed point x. If $\varepsilon = \pi/2 - r_0$, then Ω_ε coincides actually with M and we get (ii). □

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